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VARIATIONAL PRINCIPLES FOR TWO-PHASE INFILTRATION INTO A DEFORMABLE MEDIUM

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Here a method is proposed of constructing dual variational principles for two-phase infiltration into a deformable medium. The construction is based on variational treatments compiled for dissipative and elastic potentials, whose solutions are equivalent to the laws of behavior for the solid and liquid phases. The variational principles enable one to use the known porosity and saturation to determine the displacement and stress patterns in the solid phase and the pressure and velocity patterns in the liquid ones. In the case of two phases, we have variational principles for consolidation theory and two-phase infiltration.

1. Consider two-phase infiltration into a viscoplastic medium. We write [1] the equation of continuity for the solid phase

$$(1-m)_{,t} + \operatorname{div}((1-m)\dot{\mathbf{u}}) = 0; \tag{1.1}$$

the equations of continuity for the liquid phase

$$(ms)_{,t} + \operatorname{div}(ms\mathbf{v}_1) = 0; \tag{1.2}$$

$$(m(1-s))_{,t} + \operatorname{div}(m(1-s)\mathbf{v}_2) = 0; \tag{1.3}$$

the equilibrium equation

$$\sigma^f_{ij,j} - p_{,i} = 0; \tag{1.4}$$

the relation between the pressures in the liquid phases

$$p_1 - p_2 = p_c \tag{1.5}$$

and the entropy production in the energy representation for $T_1 \approx T_2 \approx T_3 \approx$ const [1]:

$$\Sigma = \sigma^f_{ij} e^p_{ij} - \mathbf{q}_1 \cdot \nabla p_1 - \mathbf{q}_2 \cdot \nabla p_2.$$

Here $\mathbf{u}$ is the vector for the solid-phase displacement; $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocities of the liquid phases; m porosity; s saturation in the first phase; $\sigma_{ij}{}^f$ the components of the tensor for the effective stresses $\boldsymbol{\sigma}^f$; $p = sp_1 + (1-s)p_2$ the mean pressure; p_1 and p_2 the pressures in the liquid phases; $p_c = p_c(s)$ the capillary pressure step; $e_{ij}{}^p = (1/2)(\dot{u}_{i,j} + \dot{u}_{j,i})$ the components of the tensor for the rates of the viscoplastic strain $\mathbf{e}^p$; $\mathbf{q}_1 = ms(\mathbf{v}_1 - \dot{\mathbf{u}})$, $\mathbf{q}_2 = m(1-s)(\mathbf{v}_2 - \dot{\mathbf{u}})$ the phase infiltration rates; and T_1, T_2, T_3 the absolute temperatures in the phases.

We introduce the symbols $\mathbf{X}_1 = -\nabla p_1$, $\mathbf{X}_2 = -\nabla p_2$, $\mathbf{X}_3 = \boldsymbol{\sigma}^f$, $\mathbf{Y}_1 = \mathbf{q}_1$, $\mathbf{Y}_2 = \mathbf{q}_2$, $\mathbf{Y}_3 = \mathbf{e}^p$ ($\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ for the generalized forces and $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)$ for the generalized velocities. To close system (1.1)-(1.5) we use the normal dissipation hypothesis [2, 3], on which there is a dissipation potential $\varphi(\mathbf{Y})$ and a convex semicontinuous eigenfunctional from below such that

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$$\mathbf{X} \in \partial\varphi(\mathbf{Y}) \tag{1.6}$$

($\mathbf{X}$ is the subgradient of $\varphi(\mathbf{Y})$ at point $\mathbf{Y}$). From (1.6) we have [3] the inverse relation

$$\mathbf{Y} \in \partial\varphi^*(\mathbf{X}), \tag{1.7}$$

in which $\varphi^*(\mathbf{X})$ is the conjugate dissipation potential, which is related to $\varphi(\mathbf{Y})$ by a Young-Fenchel transformation [4]. It has been shown [3] that the following assertions are equivalent:

$$\mathbf{X}' \in \partial\varphi(\mathbf{Y}'); \tag{1.8}$$

$$\varphi(\mathbf{Y}) - \mathbf{X}' \cdot \mathbf{Y} \tag{1.9}$$

produces the minimum with respect to $\mathbf{Y}$ at the point $\mathbf{Y} = \mathbf{Y}'$;

$$\mathbf{Y}' \in \partial\varphi^*(\mathbf{X}'); \tag{1.10}$$

$$\varphi(\mathbf{X}) - \mathbf{X} \cdot \mathbf{Y}' \tag{1.11}$$

produces the minimum with respect to $\mathbf{X}$ at the point $\mathbf{X} = \mathbf{X}'$.

Formulas (1.8)-(1.11) are the basis for the variational principles. We assume that the dissipation consists of three independent dissipative mechanisms [2]:

$$\varphi(\mathbf{Y}) = \Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2) + \Psi_3(\mathbf{e}^p); \tag{1.12}$$

$$\varphi^*(\mathbf{X}) = \Phi_1(\nabla p_1) + \Phi_2(\nabla p_2) + \Phi_3(\boldsymbol{\sigma}^f). \tag{1.13}$$

Here $\Psi_i(\cdot)$, $\Phi_i(\cdot)$ $(i = 1, 2)$ are the dissipative and conjugate dissipative potentials of the liquid phases [5], while $\Psi_3(\cdot)$, $\Phi_3(\cdot)$ are the dissipative and conjugate dissipative potentials for the viscoplastic skeleton [6]. We assume that the functionals $\varphi(\mathbf{Y})$, $\varphi^*(\mathbf{X})$ are smooth:

$$\mathbf{X} = \operatorname{grad}\varphi(\mathbf{Y}),\ \mathbf{Y} = \operatorname{grad}\varphi^*(\mathbf{X}), \tag{1.14}$$

although the subsequent results are correct for relations of the more general form (1.6) and (1.7). Certain transformations based on (1.12)-(1.14) convert (1.1)-(1.7) to

$$-p_{1,i} = \partial\Psi_1(q_1)/\partial q_{1i} \quad \text{or} \quad q_{1i} = -\partial\Phi_1(\nabla p_1)/\partial p_{1,i}; \tag{1.15}$$

$$-p_{2,i} = \partial\Psi_2(q_2)/\partial q_{2i} \quad \text{or} \quad q_{2i} = -\partial\Phi_2(\nabla p_2)/\partial p_{2,i}; \tag{1.16}$$

$$\sigma^f_{ij} = \partial\Psi_3(e^p)/\partial e^p_{ij} \quad \text{or} \quad e^p_{ij} = \partial\Phi_3(\sigma^f)/\partial\sigma^f_{ij}; \tag{1.17}$$

$$\sigma^f_{ij,j} - p_{,i} = 0; \tag{1.18}$$

$$\operatorname{div}(\dot{\mathbf{u}} + \mathbf{q}_1 + \mathbf{q}_2) = 0; \tag{1.19}$$

$$p_1 - p_2 = p_c; \tag{1.20}$$

$$m_{,t} = \operatorname{div}((1 - m)\dot{\mathbf{u}}); \tag{1.21}$$

$$-(ms)_{,t} = \operatorname{div}(\mathbf{q}_1 + ms\dot{\mathbf{u}}). \tag{1.22}$$

2. We construct a variational principle on the variables $\dot{\mathbf{u}}, \mathbf{q}_1, \mathbf{q}_2$; (1.8) and (1.9) imply that the process $(\mathbf{X}^0, \mathbf{Y}^0)$ actually occurring in region Ω will have the $\mathbf{Y}^0$ corresponding to $\mathbf{X}^0$ determined from the solution to

$$\inf_{\mathbf{Y}} B_1^0(\mathbf{Y}) = \inf_{\mathbf{Y}} \int_\Omega [\varphi(\mathbf{Y}) - \mathbf{X}^0 \cdot \mathbf{Y}]\, d\Omega. \tag{2.1}$$

The result is unaltered if the functional $B_1{}^0(\mathbf{Y})$ is minimized with respect to the variables $\dot{\mathbf{u}}$, $\mathbf{q}_1$, $\mathbf{q}_2$. In that formulation, it is trivial to solve (2.1), since it is necessary to know the forces $\mathbf{X}_1^0, \mathbf{X}_2^0, \mathbf{X}_3^0$ throughout region Ω. We transform $\int_\Omega \mathbf{X}^0 \cdot \mathbf{Y} d\Omega$, in such a way that the solution to (2.1) can be obtained simply from knowing $\mathbf{X}^0$ at the boundary Γ of region Ω. We get

$$\int_\Omega \mathbf{X}^0 \cdot \mathbf{Y} d\Omega = \int_\Omega \left(-\mathbf{q}_1 \cdot \nabla p_1^0 - \mathbf{q}_2 \cdot \nabla p_2^0 + e^p_{ij}\sigma^{f0}_{ij}\right) d\Omega = -\int_\Omega \mathbf{q}_1 \cdot \nabla((1 - s)p_c)\, d\Omega +$$

$$+\int_\Omega \mathbf{q}_2\cdot\nabla(sp_c)\,d\Omega+\int_\Gamma \Pi_i^0\dot{u}_i\,d\Gamma-\int_\Gamma q_n p^0\,d\Gamma+\int_\Omega p^0\,\mathrm{div}\,(\dot{\mathbf{u}}+\mathbf{q}_1+\mathbf{q}_2)\,d\Omega,$$

in which $q_n = q_{1n} + q_{2n}$ is the normal component of the overall infiltration rate $\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2$; $\Pi_i = (\sigma_{ij}^f - p\delta_{ij})$. We have used (1.18) and (1.20), which are satisfied by the forces $\mathbf{X}_1^0, \mathbf{X}_2^0, \mathbf{X}_3^0$. Subject to (1.19) and

$$\dot{u}_i = \dot{u}_i^0 \text{ on } \Gamma_u; \tag{2.2}$$

$$q_n = q_n^0 \text{ on } \Gamma_q \tag{2.3}$$

we pass from (2.1) to

$$\inf_{\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2\in(1.19),(2.2),(2.3)} I_1(\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2); \tag{2.4}$$

$$I_1(\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2)=\int_\Omega[\Psi_1(\mathbf{q}_1)+\Psi_2(\mathbf{q}_2)+\Psi_3(\mathbf{e}^p)]\,d\Omega+\int_\Omega \mathbf{q}_1\cdot\nabla((1-s)p_c)\,d\Omega-\int_\Omega \mathbf{q}_2\cdot\nabla(sp_c)\,d\Omega-\int_{\Gamma_\sigma}\Pi_i^0\dot{u}_i\,d\Gamma+\int_{\Gamma_p} q_n p^0\,d\Gamma,$$

$$\Gamma_\sigma+\Gamma_u=\Gamma,\ \Gamma_p+\Gamma_q=\Gamma. \tag{2.5}$$

As the variation $\delta I_1(\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2)$ is equal to zero subject to the constraints of (1.19), (2.2), and (2.3), we have that system (1.15)-(1.20) is obeyed along with the boundary conditions

$$\Pi_i = \Pi_i^0 \text{ on } \Gamma_\sigma; \tag{2.6}$$

$$p = p^0 \text{ on } \Gamma_p. \tag{2.7}$$

3. We construct a variational principle on the variables $\boldsymbol{\sigma}^f$ and p. It follows from (1.10) and (1.11) that for the process $(\mathbf{X}^0, \mathbf{Y}^0)$ actually occurring in region Ω, the $\mathbf{X}^0$ corresponding to $\mathbf{Y}^0$ is defined by the solution to

$$\inf_{\mathbf{X}} B_2^0(\mathbf{X})=\inf_{\mathbf{X}}\int_\Omega[\varphi^*(\mathbf{X})-\mathbf{X}\cdot\mathbf{Y}^0]\,d\Omega. \tag{3.1}$$

The result is unaltered if the functional $B_2{}^0(\mathbf{X})$ is minimized with respect to the variables $\boldsymbol{\sigma}^f$, p_1, p_2, and we make the substitutions $p_1 = p + (1 - s)p_c$, $p_2 = p - sp_c$, to get

$$B_2^0(\boldsymbol{\sigma}^f,\nabla p)=\int_\Omega[\Phi_1(\nabla(p+(1-s)p_c))+\Phi_2(\nabla(p-sp_c))+\Phi_3(\boldsymbol{\sigma}^f)]\,d\Omega+$$
$$+\int_\Omega\left(\mathbf{q}^0\cdot\nabla p-e_{ij}^{p0}\sigma_{ij}^f\right)d\Omega\quad\left(B_2^0(\boldsymbol{\sigma}^f,\nabla p)=B_2^0(\boldsymbol{\sigma}^f,\nabla p_1,\nabla p_2)+\text{const}\right). \tag{3.2}$$

We transform the last integral on the right in (3.2) to get

$$\int_\Omega\left(\mathbf{q}^0\cdot\nabla p-e_{ij}^{p0}\sigma_{ij}^f\right)d\Omega=\int_\Gamma\Pi_i\dot{u}_i^0\,d\Gamma-\int_\Gamma q_n^0 p\,d\Gamma-\int_\Omega\dot{u}_i^0\left(\sigma_{ij,j}^f-p_{,i}\right)d\Omega.$$

Here we have used (1.19), which is satisfied by the velocities $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$. Subject to (1.18), (2.6), and (2.7), we get from (3.1) that

$$\inf_{\boldsymbol{\sigma}^f,p\in(1.18),(2.6),(2.7)} I_2(\boldsymbol{\sigma}^f,p), \tag{3.3}$$

in which

$$I_2(\boldsymbol{\sigma}^f,p)=\int_\Omega[\Phi_1(\nabla(p+(1-s)p_c))+\Phi_2(\nabla(p-sp_c))+\Phi_3(\boldsymbol{\sigma}^f)]\,d\Omega-$$
$$-\int_{\Gamma_u}\Pi_i\dot{u}_i^0\,d\Gamma+\int_{\Gamma_q}q_n^0 p\,d\Gamma. \tag{3.4}$$

As the variation $\delta I_2(\boldsymbol{\sigma}^f, p)$ is zero subject to the constraints (1.18), (2.6), and (2.7), it follows that system (1.15)-(1.20) is obeyed together with the boundary conditions (2.2) and (2.3). We have thus obtained the variational principles (2.4) and (3.3), which are equivalent to solving the system (1.15)-(1.20) with the boundary conditions (2.2), (2.3), (2.6), (2.7) with given saturation and porosity patterns.

4. We apply the duality method [4] to get

$$\inf_{\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2\in(1.19),(2.2),(2.3)} I(\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2)=\sup_{\sigma^f,p\in(1.18),(2.6),(2.7)}[-I_2(\sigma^f,p)].$$

We transfer from the (2.4) treatment to the one dual to it on one or two variables to get six minimax treatments $\inf\sup I_i(\cdot)$, $i=\overline{3,8}$. Here we used the boundary conditions

$$q_{1n}=q_{1n}^0,\ q_{2n}=q_{2n}^0 \text{ on } \Gamma_q. \tag{4.1}$$

The functionals $I_i(\cdot)$ $(i=\overline{3,8})$ are derived in a somewhat different way. We put

$$\operatorname{div}\mathbf{q}_1=\varphi_1(\mathbf{r},t),\ \operatorname{div}\mathbf{q}_2=\varphi_2(\mathbf{r},t),\ \operatorname{div}\dot{\mathbf{u}}=\varphi_3(\mathbf{r},t), \tag{4.2}$$

and split up the minimization of (2.4) with respect to the variables $\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2$:

$$\begin{aligned}&\inf_{\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2\in(1.19),(2.2),(2.3)} I_1(\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2)=\\&=\inf_{\mathbf{q}_1\in(4.1),(4.2)}J_1(\mathbf{q}_1)+\inf_{\mathbf{q}_2\in(4.1),(4.2)}J_2(\mathbf{q}_2)+\inf_{\dot{\mathbf{u}}\in(2.2),(4.2)}J_3(\dot{\mathbf{u}}).\end{aligned} \tag{4.3}$$

Here

$$J_1(\mathbf{q}_1)=\int_\Omega[\Psi_1(\mathbf{q}_1)+\mathbf{q}_1\cdot\nabla((1-s)p_c)]\,d\Omega+\int_{\Gamma_p}q_{1n}p^0d\Gamma;$$

$$J_2(\mathbf{q}_2)=\int_\Omega[\Psi_2(\mathbf{q}_2)-\mathbf{q}_2\cdot\nabla(sp_c)]\,d\Omega+\int_{\Gamma_p}q_{2n}p^0d\Gamma;$$

$$J_3(\dot{\mathbf{u}})=\int_\Omega\Psi_3(\mathbf{e}^p)\,d\Omega-\int_{\Gamma_\sigma}\Pi_i^0\dot{u}_id\Gamma.$$

The functionals $J_4(p)$, $J_5(p)$, $J_6(\sigma^f,p)$ in

$$\sup_{p\in(2.7)}[-J_4(p)]=\inf_{\mathbf{q}_1\in(4.1),(4.2)}J_1(\mathbf{q}_1),\quad \sup_{p\in(2.7)}[-J_5(p)]=\inf_{\mathbf{q}_2\in(4.1),(4.2)}J_2(\mathbf{q}_2),$$

$$\sup_{\sigma^f,p\in(1.19),(2.6)}[-J_6(\sigma^f,p)]=\inf_{\dot{\mathbf{u}}\in(2.2),(4.2)}J_3(\dot{\mathbf{u}})$$

take the form

$$J_4(p)=\int_\Omega\Phi_1(\nabla(p+(1-s)p_c))\,d\Omega+\int_{\Gamma_q}q_{1n}^0pd\Gamma-\int_\Omega p\varphi_1d\Omega,$$

$$J_5(p)=\int_\Omega\Phi_2(\nabla(p-sp_c))\,d\Omega+\int_{\Gamma_q}q_{2n}^0pd\Gamma-\int_\Omega p\varphi_2d\Omega,$$

$$J_6(\sigma^f,p)=\int_\Omega\Phi_3(\sigma^f)\,d\Omega-\int_{\Gamma_u}\Pi_i\dot{u}_i^0d\Gamma-\int_\Omega p\varphi_3d\Omega.$$

We introduce the Lagrange multiplier $\lambda=-p$ to write the functionals:

$$J_1'(\mathbf{q}_1,p)=J_1(\mathbf{q}_1)-\int_\Omega p(\operatorname{div}q_1-\varphi_1)\,d\Omega,$$

$$J_2'(\mathbf{q}_2,p)=J_2(\mathbf{q}_2)-\int_\Omega p(\operatorname{div}q_2-\varphi_2)\,d\Omega,$$

$$J_3'(\dot{\mathbf{u}},p)=J_3(\dot{\mathbf{u}})-\int_\Omega p(\operatorname{div}\dot{u}-\varphi_3)\,d\Omega.$$

We combine the functionals J_1', J_2', J_3', J_4, J_5, J_6 in such a way as to eliminate φ_1, φ_2, and φ_3 to get the functionals $I_i(\cdot)$ $(i=\overline{1,8})$. For example, the functional $I_3(\dot{\mathbf{u}},p)$ is as follows in the variables $\dot{\mathbf{u}}$ and p usually employed in numerical solution of problems in consolidation theory:

$$I_3(\dot{\mathbf{u}}, p) = J_3'(\dot{\mathbf{u}}, p) - J_4(p) - J_5(p) = \int_\Omega [-\Phi_1(\nabla(p + (1-s)p_c)) -$$

$$- \Phi_2(\nabla(p - sp_c)) + \Psi_3(\mathbf{e}^p)]\, d\Omega - \int_\Omega p \operatorname{div} \dot{\mathbf{u}} d\Omega - \int_{\Gamma_\sigma} \Pi_i^0 \dot{u}_i d\Gamma - \int_{\Gamma_q} q_n^0 p d\Gamma,$$

and one has

$$\sup_{p\in(2.7)} \inf_{\dot{\mathbf{u}}\in(2.2)} I_3(\dot{\mathbf{u}}, p) = \inf_{\dot{\mathbf{u}},\mathbf{q}_1,\mathbf{q}_2\in(1.19),(2.2),(2.3)} I_1(\dot{\mathbf{u}}, \mathbf{q}_1, \mathbf{q}_2).$$

5. With linear infiltration laws

$$\mathbf{q}_1 = -\frac{kf_1(s)}{\mu_1}\nabla p_1, \quad \mathbf{q}_2 = -\frac{kf_2(s)}{\mu_2}\nabla p_2,$$

we can express $\mathbf{q}_1$ and $\mathbf{q}_2$ in terms of the overall velocity $\mathbf{q}$ and get the functional

$$I_1(\dot{\mathbf{u}}, \mathbf{q}) = \int_\Omega \left[\frac{1}{2}\frac{\mu_2}{k\varphi(s)}|\mathbf{q}|^2 - \mathbf{q}\cdot\nabla T(s)\right] d\Omega + \int_\Omega \Psi_3(\mathbf{e}^p)\, d\Omega - \int_{\Gamma_\sigma} \Pi_i^0 \dot{u}_i d\Gamma + \int_{\Gamma_p} p^0 q_n d\Gamma,$$

in which k is the absolute permeability; $f_1(s)$, $f_2(s)$ are the relative phase permeabilities; μ_1 and μ_2 are viscosities; and $\varphi(s) = f_1(s) + (\mu_1/\mu_2)f_2(s)$; $T(s) = \int_s^1 F(s)\, p_c'(s)\, ds + sp_c(s)$; $F(s) = f_1(s)/\varphi(s)$ is the Buckley–Leverett function.

The minimum in the functional $I_1(\dot{\mathbf{u}}, \mathbf{q})$ is attained on the actual velocity pattern $\dot{\mathbf{u}}, \mathbf{q}$ subject to the constraints (1.19), (2.2), (2.3). The treatment dual to this variational treatment is one for the maximum in the functional $[-I_2(\boldsymbol{\sigma}^f, p)]$, i.e.,

$$\inf_{\dot{\mathbf{u}},\mathbf{q}\in(1.19),(2.2),(2.3)} I_1(\dot{\mathbf{u}}, \mathbf{q}) = \sup_{\sigma^f,p\in(1.18),(2.6),(2.7)} I_2(\boldsymbol{\sigma}^f, p).$$

Here

$$I_2(\boldsymbol{\sigma}^f, p) = \int_\Omega \frac{1}{2}\frac{k\varphi(s)}{\mu_2}|\nabla(p - T(s))|^2 + \int_\Omega \Phi_3(\boldsymbol{\sigma}^f)\, d\Omega - \int_{\Gamma_u} \Pi_i \dot{u}_i^0 + \int_{\Gamma_q} p q_n^0 d\Gamma.$$

6. With $\Gamma_u = \Gamma$, $\Gamma_q = \Gamma$, $p_c = 0$ we have the form for (2.5)

$$I_1(\dot{\mathbf{u}}, \mathbf{q}_1, \mathbf{q}_2) = \int_\Omega [\Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2) + \Psi_3(\mathbf{e}^p)]\, d\Omega. \tag{6.1}$$

Then with $\Psi_1(\mathbf{q}_1) = D_1(\mathbf{q}_1)$, $\Psi_2(\mathbf{q}_2) = D_2(\mathbf{q}_2)$, $\Psi_3(\mathbf{e}^p) = D_3(\mathbf{e}^p)$ (D_1, D_2, and D_3 are dissipative functions), the actual process is determined by the minimum in the energy dissipation rate.

We put as follows in (1.15)-(1.22) instead of (1.17) for two-phase infiltration in an elastic medium with small deformations:

$$\varepsilon_{ij} = \partial W_\sigma(\boldsymbol{\sigma}^f)/\partial\sigma_{ij}^f, \quad \sigma_{ij}^f = \partial W_\varepsilon(\boldsymbol{\varepsilon})/\partial\varepsilon_{ij},$$

in which W_σ and W_ε are the elastic potentials and ε_{ij} are the components of the elastic-strain tensor. The functional that generalizes (6.1) is

$$I_1(\mathbf{u}, \mathbf{q}_1, \mathbf{q}_2) = \int_\Omega \left[\Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2) + \frac{W_\varepsilon(\varepsilon(t)) - W_\varepsilon(\varepsilon(t-\Delta t))}{\Delta t}\right] d\Omega. \tag{6.2}$$

This functional approximately characterizes the sum of the rates of energy accumulation and dissipation for $\Psi_1(\mathbf{q}_1) = D_1(\mathbf{q}_1)$, $\Psi_2(\mathbf{q}_2) = D_2(\mathbf{q}_2)$. In the general case, it is represented as

$$I_1(\mathbf{u}, \mathbf{q}_1, \mathbf{q}_2) = \int_\Omega \left[\Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2) + \frac{W_\varepsilon(\varepsilon(t)) - W_\varepsilon(\varepsilon(t-\Delta t))}{\Delta t}\right] d\Omega +$$

$$+ \int_\Omega \mathbf{q}_1\cdot\nabla((1-s)p_c)\, d\Omega - \int_\Omega \mathbf{q}_2\cdot\nabla(sp_c)\, d\Omega -$$

$$-\int_{\Gamma_\sigma} \Pi_i^0 \frac{u_i(t) - u_i(t-\Delta t)}{\Delta t} d\Gamma + \int_{\Gamma_p} q_n p^0 d\Gamma. \tag{6.3}$$

One gets the solution to

$$\inf I_1(\mathbf{u}, \mathbf{q}_1, \mathbf{q}_2) \tag{6.4}$$

subject to (2.2), (2.3), and

$$\operatorname{div}\left(\frac{\mathbf{u}(t) - \mathbf{u}(t-\Delta t)}{\Delta t} + \mathbf{q}_1 + \mathbf{q}_2\right) = 0$$

on the actual pattern for the variables $\mathbf{u}$, $\mathbf{q}_1$, $\mathbf{q}_2$. The dual variational principles are constructed as in two-phase infiltration into a viscoplastic medium. One can construct various forms of numerical realization for such two-phase infiltration into an elastic medium. For example, instead of (6.3) one can use

$$\begin{aligned} I_1\left(\mathbf{u}^k, \mathbf{q}_1^k, \mathbf{q}_2^k, \alpha\right) = &\int_\Omega \frac{W_\varepsilon(\varepsilon^k) - W_\varepsilon(\varepsilon^{k-1})}{\Delta t_k} - \int_{\Gamma_\sigma} \Pi_i^{0k} \frac{u_i^k - u_i^{k-1}}{\Delta t_k} d\Gamma + \\ &+ \alpha \left\{ \int_\Omega \left[\Psi_1(\mathbf{q}_1^k) + \Psi_2(\mathbf{q}_2^k) + \mathbf{q}_1^k \cdot \nabla\left((1-s^k) p_c^k\right) - \right.\right. \\ &\left.\left. - \mathbf{q}_2^k \cdot \nabla\left(s^k p_c^k\right)\right] d\Omega + \int_{\Gamma_q} q_n^k p^{0k} d\Gamma \right\}, \end{aligned} \tag{6.5}$$

where $a^k = a(t_k)$; $\Delta t_k = t_k - t_{k-1}$; $0 < \alpha \le 1$. The minimum here subject to (2.2), (2.3), and

$$\operatorname{div}\left[\frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\Delta t_k} + \alpha\left(\mathbf{q}_1^k + \mathbf{q}_2^k\right) + (1-\alpha)\left(\mathbf{q}_1^{k-1} + \mathbf{q}_2^{k-1}\right)\right] = 0$$

is attained on the actual pattern of the variables $\mathbf{u}^k$, $\mathbf{q}_1^k$, $\mathbf{q}_2^k$.

7. A decoupling similar to (4.3) occurs in the case of an elastic skeleton in (6.4) and in other cases with various law of behavior for the individual phases. Constructing variational principles amounts thus to constructing them for the individual phases.

The (4.3) representation can be considered as a splitting into two tasks, one of which characterizes the strain and the other the two-phase infiltration. This indicates how to use existing formulations in the theory of deformable solids and the theory of two-phase infiltration.

One can incorporate changes in the porosity m and saturation s by means of (1.21) and (1.22). That approach is an extension of the algorithm for solving for two-phase infiltration with separation with respect to the pressure and saturation. In the particular case of two phases, the variational principles for two-phase infiltration into a deformable medium give variational principles for consolidation and two-phase infiltration theory [7].

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